TRIBONACCI CORDIAL LABELING OF PATH & CYCLE RELATED GRAPHS

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Abstract: The Tribonacci Cordial labeling is an extension of Fibonacci Cordial labeling, a well-known forms of vertex labelings. An injective function $f: V(G) \to \{T_0, T_1, \dots, T_n\}$ is said to be Tribonacci Cordial labeling if the induced function $f^*: E(G) \to \{0,1\}$ defined by $f^*(uv) = (f(u) + f(v))(mod\ 2)$ satisfies the conditions $|e_f(0) - e_f(1)| \leq 1$. A Graph which admits Tribonacci Cordial labeling is called Tribonacci Cordial Graph. In this paper, we investigate whether some well-known Graphs are Tribonacci Cordial.

Keywords: Tribonacci Cordial labeling, Star Graph, Coconut Tree Graph, Total Path Graph, Quadrilateral Snake Graph, Jewel Graph.

I INTRODUCTION

Graph labeling is a significant area of research in discrete mathematics, with applications in network design, cryptography, coding theory and combinatorial optimization. It involves assigning numbers to the vertices or edges of a graph with specific conditions. Cordial labeling was introduced by Cahit in 1987[1], ensures a balanced distribution of assigning labels. In this paper, we study Tribonacci Cordial labeling for different Graphs, a variation of Cordial labeling that incorporates the Tribonacci sequence, where each term is the sum of the previous three terms. An injective function $f: V(G) \to \{T_0, T_1, \dots, T_n\}$ is said to be Tribonacci Cordial labeling if the induced function $f^*: E(G) \to \{0,1\}$ defined by $f^*(uv) = \big(f(u) + f(v)\big)(mod\ 2)$ satisfies the conditions $|e_f(0) - e_f(1)| \leq 1$. A Graph which admits Tribonacci Cordial labeling is called Tribonacci Cordial Graph. In this study, we examine the existence of Tribonacci Cordial labeling for Star Graph, Coconut Tree Graph, Total Path Graph, Quadrilateral Snake Graph and Jewel Graph.

II PRELIMINARIES

In this section, we recall some basic definitions related to this paper. In this paper, the Graphs considered are all simple, connected and undirected.

Definition 2.1: A **Graph** G consists of a pair $(V(G), X(G))$ where $V(G)$ is a non-empty finite set whose elements are called points or vertices and $X(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called lines or edges of the Graph G.

Definition 2.2: A **Path** is a sequence of vertices where each consecutive pair of vertices is connected by an egde in the Graph.

Definition 2.3: A **Cycle** is a non-empty path in which the first and last vertices are identical.

Definition 2.4: A **Tree** is an undirected Graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected Graph.

Definition 2.5: Total Graph $T(G)$ is a Graph with the vertex set $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 2.6: A **Star Graph** S_n is a Tree where one vertex is connected to all other vertices.

Definition 2.7: A **Coconut Tree Graph** $CT(m, n)$ is obtained from the path P_m by appending n new pendent edges at an end vertex of P_m.

Definition 2.8: Total Path Graph $T(P_m)$ has $2m - 1$ vertices and $4m - 5$ edges, the vertex set and the edge set of $T(P_m)$ be

$V(T(P_m)) = \{v_i, 0 \leq i \leq m-1\} \cup \{u_i / 0 \leq i \leq m-2\}$ and

$E(T(P_m)) = \{v_i v_{i+1} / 0 \leq i \leq m-2\} \cup \{u_i u_{i+1} / 0 \leq i \leq m-3\} \cup \{u_i v_{i+1} / 0 \leq i \leq m-2\} \cup \{u_i v_i / 0 \leq i \leq m-2\}$.

Definition 2.9: The **Fish Graph** is a planer undirected Graph with 6 vertices and 7 edges in the form of a cycles (C_3, C_4) and adjoined by a common vertex.

Definition 2.10:[8] A **Quadrilateral Snake Graph** $\boldsymbol{Q_n}$ is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to new vertices u_i and w_i for $1 \leq i \leq n-1$ respectively and then joining u_i and w_i.

Definition 2.11: The **Jewel Graph** J_n is obtained from a cycle C_4 by adding a prime edge connecting two non-adjacent vertices of the cycle and also appending an edge from two vertices of the cycle C_4, meeting at a common vertex.

Definition 2.12:[3] The **Bull Graph** is a planer undirected Graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendant edges attached to disjoint vertices.

Definition 2.13: The **Cricket Graph** is a planer undirected Graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendant vertices attached to same vertex.

Definition 2.14: Graph labeling is an assignment of integers to vertices or edges, or both, under certain conditions.

Definition 2.15:[7] A function $f: V(G) \to \{0,1\}$ is said to be **Cordial Labeling** if the induced function $f^*: E(G) \to \{0,1\}$ defined by $f^*(uv) = |f(u) - f(v)|$ satisfies the conditions $|v_f(0) - v_f(1)| \le 1$, as well as $|e_f(0) - e_f(1)| \le 1$, where,

$v_f(0) =$ Number of vertices with label 0,

$v_f(1) =$ Number of vertices with label 1,

$e_f(0) =$ Number of edges with label 0,

$e_f(1) =$ Number of edges with label 1.

Definition 2.16:[7] The Tribonacci Sequence is defined by each term is the sum of the three preceding terms. In other words, The sequence T_n of Tribonacci numbers is defined by the third order linear recurrence relation $(for\, n \ge 0)$: $T_{n+3} = T_n + T_{n+1} + T_{n+2}; T_0 = 0,\ T_1 = T_2 = 1$ i.e., $\{0,1,1,2,4,7,13,24,44,81, \dots\}$ is the **Tribonacci sequence.**

Definition 2.17:[7] An injective function $f: V(G) \to \{T_0, T_1, \dots, T_n\}$ is said to be **Tribonacci Cordial Labeling** if the induced function $f^*: E(G) \to \{0,1\}$ defined by $f^*(uv) = (f(u) + f(v))(mod\ 2)$ satisfies the conditions $|e_f(0) - e_f(1)| \le 1$. A Graph which admits Tribonacci Cordial labeling is called Tribonacci Cordial Graph.

III TRIBONACCI CORDIAL LABELING GRAPHS

Theorem 3.1: The Star Graph S_n is a Tribonacci cordial Graph.

Proof: A Star Graph is a tree with one internal node and multiple leaves. Let v be the central vertex. Let $\{v_1, v_2, \dots, v_n\}$ be the pendant vertices adjacent to v.

The resultant Graph is S_n with

$V(S_n) = \{v\} \cup \{v_i / 1 \le i \le n-1\}$ and

$E(S_n) = \{vv_i / 1 \le i \le n-1\}$.

Define a function $f: V(S_n) \to \{T_0, T_1, \dots, T_n\}$ by

$$f(v) = T_{n-1}$$

$$f(v_i) = T_{i-1}, 1 \le i \le n-1$$

$$f^*(uv) = (f(u) + f(v))(mod\ 2)$$

Example for Tribonacci cordial labeling for the Graph S_9 is given in the Figure 3.1

Edge label calculation for S_9 is given by the following equations

$f^*(uv) = (f(u) + f(v))(mod\ 2)$

$f^*(vv_1) = (f(v) + f(v_1))(mod\ 2) = T_8 + T_0(mod\ 2) = 44 + 0(mod\ 2) = 0$

$f^*(vv_2) = (f(v) + f(v_2))(mod\ 2) = T_8 + T_1(mod\ 2) = 44 + 1(mod\ 2) = 1$

$f^*(vv_3) = (f(v) + f(v_3))(mod\ 2) = T_8 + T_2(mod\ 2) = 44 + 1(mod\ 2) = 1$

$f^*(vv_4) = (f(v) + f(v_4))(mod\ 2) = T_8 + T_3(mod\ 2) = 44 + 2(mod\ 2) = 0$

$f^*(vv_5) = (f(v) + f(v_5))(mod\ 2) = T_8 + T_4(mod\ 2) = 44 + 4(mod\ 2) = 0$

$f^*(vv_6) = (f(v) + f(v_6))(mod\ 2) = T_8 + T_5(mod\ 2) = 44 + 7(mod\ 2) = 1$

$f^*(vv_7) = (f(v) + f(v_7))(mod\ 2) = T_8 + T_6(mod\ 2) = 44 + 13(mod\ 2) = 1$

$f^*(vv_8) = (f(v) + f(v_8))(mod\ 2) = T_8 + T_7(mod\ 2) = 44 + 24(mod\ 2) = 0$

$|e_f(0) - e_f(1)| = |4 - 4| = 0 \leq 1$

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Therefore, Star Graph S_n admits Tribonacci Cordial labeling for all $n \geq 3$.

Theorem 3.2: The Coconut Tree Graph $CT(m,n)$ is a Tribonaaci Cordial Graph.

Proof: Let v_i, $1 \leq i \leq m$ be the vertices of the path P_m and let u_i, $1 \leq i \leq n$ be the new pendant vertices which are attached to the m^{th} vertex of the path P_m.

The resultant Graph is $CT(m,n)$ whose vertex set is

$V[CT(m,n)] = \{\{v_i/1 \leq i \leq m\} \cup \{u_i/1 \leq i \leq n\}\}$ and edge set is

$E[CT(m,n)] = \{\{v_iv_{i+1}/1 \leq i \leq m-1\} \cup \{v_mu_i/1 \leq i \leq n\}\}$ such that

$|V[CT(m,n)]| = p = m + n$ and $|E[CT(m,n)]| = q = m + n - 1$.

Define a function $f: V(G) \to \{T_0, T_1, \dots, T_n\}$.

In order to assign Tribonacci Cordial labeling to the Graph $CT(m,n)$, the labeling is given under two cases.

Case (i): If $m = n$. The vertex and edge labeling is given by the following equation.

$f(v_i) = T_{i-1}, 1 \leq i \leq m$

$f(u_i) = T_{n-1+i}, 1 \leq i \leq n$

Case (ii): If $m \neq n$.The vertex and edge labeling is given by the following equation.

$f(v_m) = T_{m+n-1}$

$f(v_i) = T_{i-1}, 1 \leq i \leq m-1$

$f(u_i) = T_{(m-2)+i}, 1 \leq i \leq n$

Example for Tribonacci Cordial labeling for the Graph $CT(5,5)$ and $CT(3,7)$ is shown in the Figure 3.2(a) and Figure 3.2(b) for case (i) & case (ii) respectively.

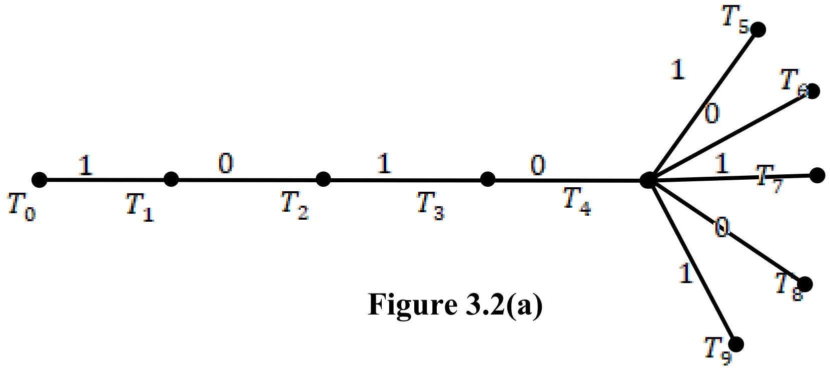

Figure 3.2(a)

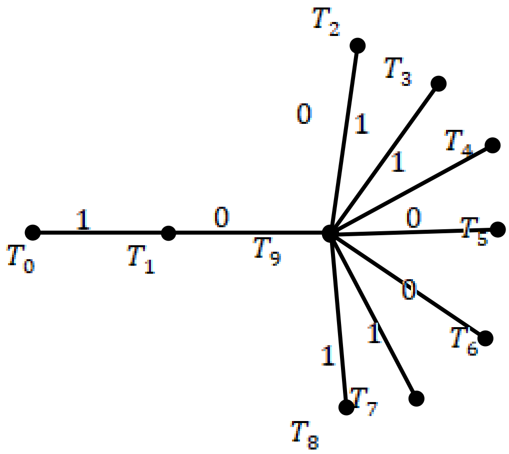

Figure 3.2(b)

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Hence we conclude that the Coconut Tree Graph $CT(m, n)$ admits Tribonacci Cordial labeling for all $m \geq 2, n \geq 1$.

Theorem 3.3: Total Path Graph $T(P_m)$ is a Tribonacci Cordial Graph.

Proof: Let $V(T(P_m)) = \{v_i, 0 \leq i \leq m-1\} \cup \{u_i/0 \leq i \leq m-2\}$

Let $E(T(P_m)) = \{v_i v_{i+1}/0 \leq i \leq m-2\} \cup \{u_i u_{i+1}/0 \leq i \leq m-3\} \cup \{u_i v_{i+1}/0 \leq i \leq m-2\} \cup \{u_i v_i/0 \leq i \leq m-2\}$

The cardinality of the vertex set and the edge set of $T(P_m)$ are

$|V(T(P_m)| = p = 2m-1$ and $|E(T(P_m)| = q = 4m-5$.

Define a function $f: V(G) \to \{T_0, T_1, \ldots, T_n\}$ by

$$f(v_i) = \begin{cases} T_{2i}, & if\ i \equiv 0(mod\ 2); \\ T_{2i+1}, & if\ i \equiv 1(mod\ 2), \end{cases}$$

$$f(u_i) = \begin{cases} T_{2i+1}, & if\ i \equiv 0(mod\ 2); \\ T_{2i}, & if\ i \equiv 1(mod\ 2), \end{cases}$$

Example for Tribonacci Cordial labeling for the Graph $T(P_6)$ is given in the Figure 3.3

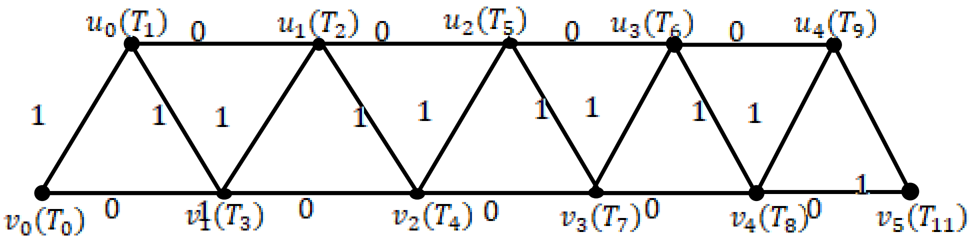

Figure 3.3

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Therefore, Total Graph of Path $T(P_m)$ admits Tribonacci Cordial labeling.

Theorem 3.4: Fish Graph is a Tribonacci Cordial Graph.

Proof: Let $V(G) = \{v_i/1 \leq i \leq 6\}$ be the vertices of C_3 and C_4 which are adjoined by a common vertex v_3 which results a Fish Graph such that $|V(G)| = p = 6$ and $|E(G)| = q = 7$. The following function assigns Tribonacci Cordial labeling to the vertices of the Fish Graph.

Define a function $f: V(G) \rightarrow \{T_0, T_1, \dots, T_n\}$ by $f(v_i) = T_{i-1}, 1 \leq i \leq 6$.

Tribonacci Cordial labeling for Fish Graph is given in the Figure 3.4

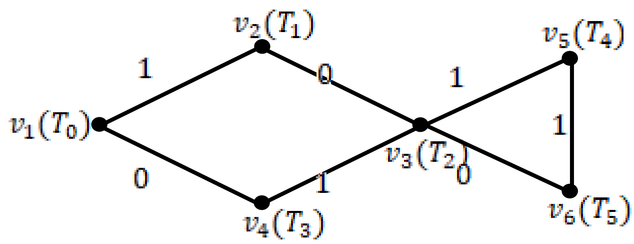

Figure 3.4

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Therefore, Fish Graph admits Tribonacci Cordial labeling.

Theorem 3.5: The Quadrilateral Snake Graph Q_n is a Tribonacci Cordial Graph.

Proof: Let $\{v_i/0 \leq i \leq n-1\}$ be the vertices of the path P_n and

Let $\{u_i, w_i / 1 \leq i \leq n-1\}$ be the new vertices which are joined to each vertices of the path P_n. The resultant Graph is Q_n whose vertex set is

$V(Q_n) = \{\{v_i/1 \leq i \leq n\} \cup \{u_i, w_i/1 \leq i \leq n-1\}\}$

$E(Q_n) = \{\{v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{u_i w_i/1 \leq i \leq n-1\} \cup \{v_i u_i/1 \leq i \leq n-1\} \cup \{v_i w_{i-1}/2 \leq i \leq n\}\}$

such that

$|V(Q_n)| = p = 3n-2$ and $|E(Q_n)| = q = 4n-4$.

Define a function $f: V(G) \to \{T_0, T_1, \dots, T_n\}$ by

$$f(v_1) = T_0$$

$$f(u_1) = T_3$$

$$f(w_1) = T_2$$

$$f(v_i) = T_{3i-5}, 2 \leq i \leq n$$

$$f(u_i) = T_{3i-1}, 2 \leq i \leq n-1$$

$$f(w_i) = T_{3i}, 2 \leq i \leq n-1$$

Example for Tribonacci Cordial labeling for the Graph Q_7 is givenin the Figure 3.5

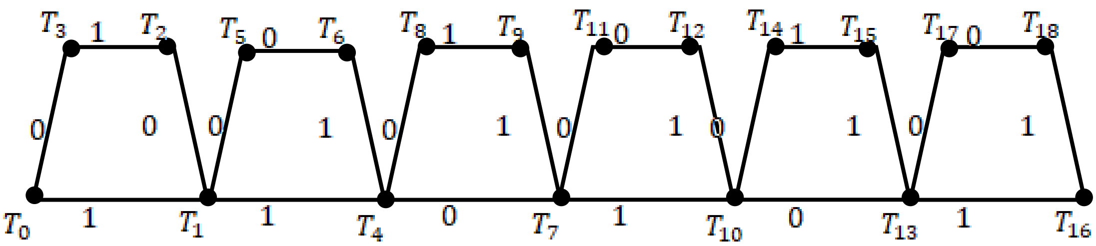

Figure 3.5

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Therefore, Quadrilateral Snake Graph Q_n admits Tribonacci Cordial labeling for all $n \geq 2$.

Theorem 3.6: Jewel Graph J_n is a Tribonacci Cordial Graph.

Proof: Let u, v, x, y be the vertices of C_4 formed by joining x and y with a prime edge and also by appending an edge from u and v which meets at common vertices $v_i, 1 \leq i \leq n$. The resultant Graph is J_n whose vertex set is

$V(J_n) = \{u, v, x, y, v_i/1 \leq i \leq n\}$ and edge set is

$E(J_n) = \{ux, uy, vx, vy, xy, uv_i, vv_i/1 \leq i \leq n\}$ such that

$|V(J_n)| = p = n+4$ and $|E(J_n)| = q = 2n+5$.

Define a function $f: V(G) \to \{T_0, T_1, \dots, T_n\}$ by

$f(u) = T_0, f(x) = T_1, f(v) = T_2, f(y) = T_3, f(v_i) = T_{3+i}, 1 \leq i \leq n.$

Example for Tribonacci Cordial labeling for the Graph J_3 is given in the Figure 3.6

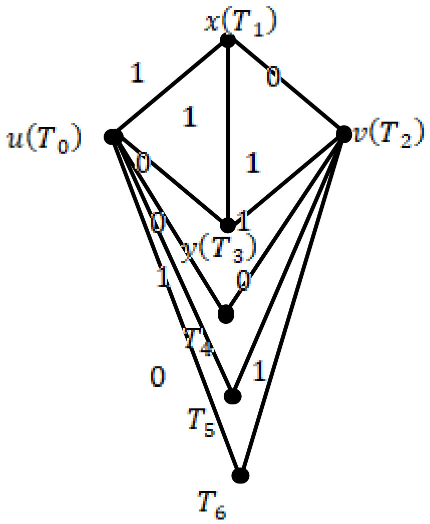

Figure 3.6

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Therefore, Jewel Graph Q_n admits Tribonacci Cordial labeling for all $n \geq 1$.

Theorem 3.7: Bull Graph is a Tribonacci Cordial Graph.

Proof: Let $V(G) = \{v_i / 1 \leq i \leq 5\}$ be the vertices and $E(G) = \{e_i / 1 \leq i \leq 5\}$ be the edges of G which is in the form of triangle with two disjoint pendant edges. Let v_2, v_3, v_4 be the vertices of the triangle and v_1, v_5 be the pendant vertices adjacent to v_2, v_4 which results a Bull Graph. The following function assigns Tribonacci Cordial labeling to the vertices of the Bull Graph, $f(v_i) = T_{i-1}, 1 \leq i \leq 5$.

Tribonacci Cordial labeling for Bull Graph is given in the Figure 3.7

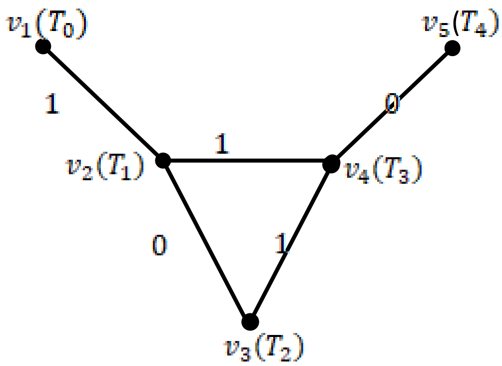

Figure 3.7

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Therefore, Bull Graph admits Tribonacci Cordial labeling.

Remark 3.8: Cricket Graph, which has the same vertex set $V(G)$ and edge set $E(G)$ with same labeling as Bull Graph is Tribonacci Cordial Graph.

IV CONCLUSION

In this study, we explored Tribonacci Cordial labeling for various Graph structures. The findings contribute to the broader field of Graph labeling, particularly in the application of Tribonacci sequences. The successful labeling of these Graphs suggests that Tribonacci Cordial labeling can be extended to other families of Graphs, paving the way for future research. The main objective was to determine whether the given Graphs admits a Tribonacci Cordial labeling and analyze their properties in relation to Tribonaaci numbers.

REFERENCES

[1] Cahit .I, Cordial Graphs: A weaker version of graceful and harmonious Graphs, Ars Combinatoria, 23 (1987), 201–207.

[2] Gallian .J.A, A Dynamic Survey of Graph Labeling, Article number #DS6, Electronic Journal of Combinatorics 1 (2013).

[3] Poulomy Ghosh, Sachchida N, Mishra, Anita pal, Various labelling on Bull graph and some related graphs, International ,Journal of Application of Fuzzy Sets and artificial intelligence (ISSN 2241-1240) vol:5 (2015) 23-35.

[4] Rokad .A.H, Fibonacci Cordial labeling of some special Graphs, Oriental Journal of Computer Science and Technology, 10(4) (2017) 824-828.

[5] Rokad .A.H, and Ghodasara .G.V, Fibonacci Cordial labeling of some special Graphs, Annals of Pure and Applied Mathematics, 11(1) (2016) 133-144.

[6] Sarbari Mitra and Soumya Bhoumik, Fibonacci Cordial labeling of some special families of Graphs, Annals of Pure and Applied Mathematics, 21(2020) 135-140.

[7] Sarbari Mitra, Soumya Bhoumik, Tribonacci Cordial Labeling of Graphs, Journal of Applied Mathematics and physics, 2022, 10, 1394 - 1402. doi:10.4236/jamp.2022.104098.

[8] Sulayman, J. E., & Pedrano, A. C. (2023). On Fibonacci Cordial Labeling of Some Snake Graphs. European Journal of Mathematics and Statistics, 4(2), 29–35.