TRIBONACCI r-GRACEFUL LABELING OF PATH RELATED GRAPHS

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Abstract: A Tribonacci r-graceful labeling is a specific type of graph labeling where the vertex labels are derived from the Tribonacci sequence**.** Let r be any real natural number. An injective function $\phi : V(G) \to \{0,1,2,\ldots,T_{q+r-1}\}$, where T_{q+r-1} is the $(q+r-1)^{th}$ Tribonacci number in the Tribonacci sequence is said to be Tribonacci r-graceful if the induced edge labeling $\phi^*: E(G) \to \{T_r, T_{r+1}, \ldots, T_{q+r-1}\}$ such that $\phi^*(uv) = |\phi(u) - \phi(v)|$ is bijective.If a graph G admits Tribonacci r-graceful labeling, then G is called a Tribonacci r-graceful graph. A graph G is said to be Tribonacci arbitrarily graceful if it is Tribonacci r-graceful for all r.

Keywords: Tribonacci r-graceful labeling, Path graph, Comb graph, Coconut Tree graph, Star graph, Fork graph

I INTRODUCTION

Graph labeling is an important area of research in graph theory, involving the assignment of labels, usually represented by integers, to the vertices and edges of a graph while satisfying specific conditions. One such labeling is Tribonacci r-graceful labeling, which extends the concept of graceful labeling by incorporating the Tribonacci sequence. Labeling of graph is the assignment of values to vertices and edges or both subject to certain conditions. In 1967, Rosa [2] introduced the concept of graceful labeling. In 1982, Slater [3] introduced the concept of *k* graceful labeling of graphs. Let *G* be a simple graph with *p* vertices and *q* edges. Let *k* be any natural number. Define an injective mapping

$\phi : V(G) \to \{0,1,2,\ldots,T_{q+k-1}\}$, where T_{q+k-1} is the $(q+k-1)^{th}$ Tribonacci number in the Tribonacci sequence is said to be Tribonacci k-graceful if the induced edge labeling $\phi^*: E(G) \to \{T_k, T_{k+1}, \ldots, T_{q+k-1}\}$ such that $\phi^*(uv) = |\phi(u) - \phi(v)|$ is bijective. If a graph G admits Tribonacci k-graceful labeling, then G is called a Tribonacci k-graceful graph. A graph G is said to be Tribonacci arbitrarily graceful if it is Tribonacci k-graceful for all k. In this sequel, we give a new concept Tribonacci r-graceful labeling of graphs.

II PRELIMINARIES

In this section, we recall some basic definitions related to this paper.

Definition 2.1: A **Graph** G consists of a pair $(V(G), X(G))$ where $V(G)$ is non empty finite set whose elements are called points or vertices and $X(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called lines or edges of the graph G.

Definition 2.2: A **Path** is a sequence of vertices where each consecutive pair of vertices is connected by an edge in the Graph.

Definition 2.3: A **Tree** is an undirected Graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected Graph.

Definition 2.4: A **Complete graph** is a graph in which each vertex is connected to every other vertex.

Definition 2.5: The **Comb graph** $P_n \odot K_1$ is obtained by joining a single pendant edge to each vertex of the path P_n.

Definition 2.6: A **Coconut Tree graph** $CT(m,n)$ is obtained from the path P_n by appending n new pendent edges at an end vertex of P_n.

Definition 2.7: A **Star graph** S_n is a tree where one vertex is connected to all other vertices.

Definition 2.8: A **Fork graph** is a tree graph with 5 vertices and 4 edges. It is also called as chair graph

Definition 2.9: The **Tribonacci sequence** is a sequence of integers T_n defined by

$T_0 = 0$, $T_1 = T_2 = 1$, $T_n = T_{n-1} + T_{n-2} + T_{n-3}\ (n \geq 3)$.

Definition 2.10: A **Graceful labeling** of a graph $G = (V, E)$ with $m = |V|$ vertices and $n = |E|$ edges is a one to one mapping $\phi: V(G) \to \{0,1,2,..n\}$ of the vertex set into the set $\{0,1,2,\ldots,n\}$ with the following property for any edge $e\{u,v\} \in E(G)$, the value $\phi^*(e) = |\phi(u) - \phi(v)|$ then $\phi^*: E(G) \to \{1,2,..,n\}$ *is* a one to one mapping of the set $E(G)$ onto the set $\{1,2,\ldots,n\}$.

Definition 2.11: Let r be any natural number. Let r be any real natural number. An injective function $\phi : V(G) \to \{0,1,2,\ldots,T_{q+r-1}\}$, where T_{q+r-1} is the $(q+r-1)^{th}$ Tribonacci number in the Tribonacci sequence is said to be Tribonacci r-graceful if the induced edge labeling $\phi^*: E(G) \to \{T_r, T_{r+1}, \ldots, T_{q+r-1}\}$ such that $\phi^*(uv) = |\phi(u) - \phi(v)|$ is bijective.If a graph G admits Tribonacci r-graceful labeling, then G is called a **Tribonacci r-graceful graph.**

Definition 2.12: A graph G is said to be **Tribonacci arbitrarily graceful** if it is Tribonacci r-graceful for all r.

III TRIBONACCI r-GRACEFUL LABELING

Theorem 3.1: The Path graph P_n is Tribonacci r-graceful for all $n \geq 2$.

Proof: Let P_n be a path graph of length $n-1$ with vertex set $V(P_n) = \{u_i/1 \leq i \leq n\}$ and edge set $E(P_n) = \{u_i u_{i+1}/1 \leq i \leq n-1\}$ such that

$|V(P_n)| = p = n$ and $|E(P_n)| = q = n-1$

Define a function $\phi : V(P_n) \to \{0, 1,2, \ldots . . T_{q+r-1}\}$ for all k by

$\phi(u_1) = T_0$

$\phi(u_i) = \phi(u_{i-1}) + (-1)^i T_{q+r-i+1}, 2 \leq i \leq n, r \geq 1$

The example of Tribonacci 2-graceful labelling is P_5 is given in Figure 3.1

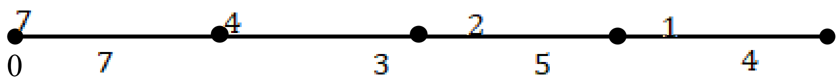

Figure 3.1

Edge label calculation for P_5 is given by the following equations

$\phi^*(uv) = |\phi(u) - \phi(v)|$

$\phi^*(u_1u_2) = |\phi(u_1) - \phi(u_2)| = |0-7| = 7$

$\phi^*(u_2u_3) = |\phi(u_2) - \phi(u_3)| = |7-3| = 4$

$\phi^*(u_3u_4) = |\phi(u_3) - \phi(u_4)| = |3-5| = 2$

$\phi^*(u_4u_5) = |\phi(u_4) - \phi(u_5)| = |5-4| = 1$

Hence the path graph $P_n, n \geq 2$ is Tribonacci r-graceful graph.

Theorem 3.2: The comb graph $P_n \odot K_1$ is Tribonacci r-graceful for all $n \geq 2$.

Proof: Let u_i, $1 \leq i \leq n$ be the vertices of the path P_n and Let v_i, $1 \leq i \leq n$ be the vertices which are joined to each vertices of the path P_n.

The resultant graph $P_n \odot K_1$ whose vertex set is

$V(P_n \odot K_1) = \{u_i, v_i/1 \leq i \leq n\}$ and edge set is

$E(P_n \odot K_1) = \{\{u_i v_i/1 \leq i \leq n\} \cup \{u_i u_{i+1}/1 \leq i \leq n-1\}\}$ such that

$|V(P_n \odot K_1)| = p = 2n$ and $|E(P_n \odot K_1)| = q = 2n-1$

Case (i): $n = 2$

Define a function $\phi : V(P_n) \to \{0,1,2, \ldots, T_{q+r-1}\}$ for all k by

$\phi(u_1) = T_0$,

$\phi(u_2) = T_{r+1}$, $r \geq 1$,

$\phi(v_1) = T_{q+r-1}$, $r \geq 1$,

$\phi(v_2) = \phi(u_2) - T_r, r \geq 1$

Case (ii): $n \geq 3$

Define a function $\phi : V(P_n) \to \{0,1,2, \dots, T_{q+r-1}\}$for all k by

$\phi(u_1) = T_0$,

$\phi(u_i) = \phi(u_{i-1}) + (-1)^i T_{q+r-i+1}, 2 \leq i \leq n, r \geq 1$

$\phi(v_1) = T_r, r \geq 1$

$\phi(v_i) = \phi(u_i) - T_{r+i-1}, 2 \leq i \leq n, r \geq 1$

The example of Tribonacci 3-graceful labelling of comb graph $P_2 \odot K_1$ for Case (i) and the Tribonacci 2-graceful labeling of comb graph $P_5 \odot K_1$ for Case (ii) is given in Figure 3.2(a) and Figure 3.2(b) respectively.

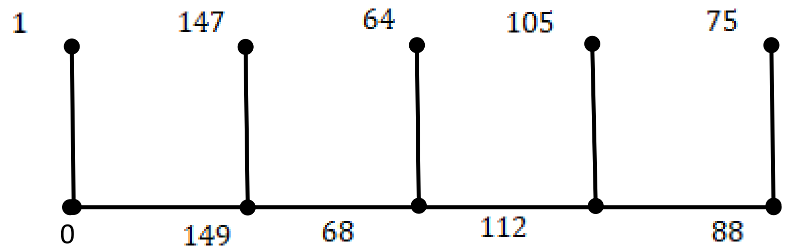

Figure 3.2 (a)

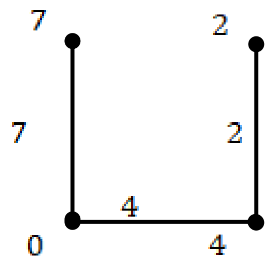

Figure 3.2 (b)

Thus ϕ admits Tribonacci r- graceful labeling for all $n \geq 2$

Hence the Comb graph $P_n \odot K_1$ is Tribonacci r-graceful for all $n \geq 2$.

Theorem 3.3: The Coconut Tree graph $CT(m, n)$ is Tribonacci r-graceful for all $m, n \geq 2$.

Proof: Let u_i, $1 \leq i \leq m$ be the vertices of the path P_m and let v_i, $1 \leq i \leq n$ be the new vertices which are attached to the m^{th} vertex of the path P_m.

The resultant graph is $CT(m,n)$whose vertex set is

$V[CT(m,n)] = \{\{u_i/1 \leq i \leq m\} \cup \{v_i/1 \leq i \leq n\}\}$ and edge set is

$E[CT(m,n)] = \{\{u_i u_{i+1}/1 \leq i \leq m-1\} \cup \{u_m v_i/1 \leq i \leq n\}\}$ such that

$|V[CT(m,n)]| = p = m + n$ and $|E[CT(m,n)]| = q = m + n - 1$

Define a function $\phi : V(P_n) \rightarrow \{0,1,2,\dots,T_{q+r-1}\}$ for all k by

$\phi(u_1) = T_0$,

$\phi(u_i) = \phi(u_{i-1}) + (-1)^i T_{q+r-i+1}, 2 \leq i \leq m, r \geq 1$

$\phi(v_i) = \phi(u_m) - T_{r+i-1}, 1 \leq j \leq n, r \geq 1$

The example of Tribonacci 1-graceful labeling of Coconut Tree graph $CT(5,5)$ is given in Figure 3.3

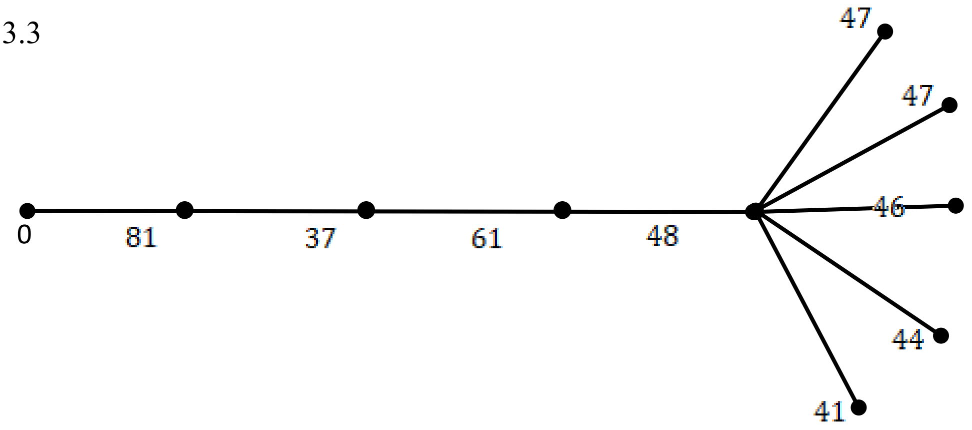

Figure 3.3

Thus ϕ admits Tribonacci graceful labelling for all r.

Hence the Coconut Tree graph $CT(m,n)$ is Tribonacci r-graceful for all $m, n \geq 2$.

Theorem 3.4: The Star graph S_n is Tribonacci r-graceful labeling for all $n \geq 3$.

Proof: A star graph is a tree with one internal node and multiple leaves. Let v be the central vertex. Let $\{v_1, v_2, \dots, v_n\}$ be the pendant vertices adjacent to v.

The resultant graph is S_n with

$V(S_n) = \{v\} \cup \{v_i/1 \leq i \leq n-1\}$ and

$E(S_n) = \{vv_i/1 \leq i \leq n-1\}$.

Define a function $\phi : V(G) \rightarrow \{0,1,2,\ldots,T_{q+r-1}\}$ by

$\phi(v) \quad = T_{q+r-1},$

$\phi(v_1) \quad = T_0,$

$\phi(v_i) \quad = \phi(v) - T_{q+r-1-(i-1)}, 2 \leq i \leq n.$

$\phi^*(uv) = |\phi(u) - \phi(v)|$

The example of Tribonacci r-graceful labelling of S_8 is given in Figure 3.4

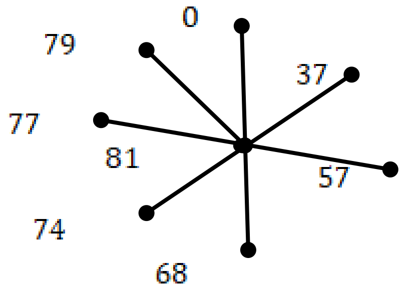

Figure 3.4

Thus ϕ admits Tribonacci graceful labeling for all r. Therefore, Star graph S_n is Tribonacci arbitrarily graceful graph.

Theorem 3.5: The Fork graph is Tribonacci arbitrarily graceful.

Proof: Let $V(G) = \{v_i / 1 \leq i \leq 5\}$

Let $E(G) = \{e_i / 1 \leq i \leq 4\}$.

Define a function $\phi : V(G) \rightarrow \{0,1,2,\ldots,T_{q+r-1}\}$ by

$\phi(v_1) \quad = T_0,$

$\phi(v_2) \quad = T_{r+1},$

$\phi(v_3) \quad = T_{q+r-1},$

$\phi(v_4) = \phi(v_2) - T_r,$

$\phi(v_5) = \phi(v_2) + T_{r+2},$

$\phi^*(uv) = |\phi(u) - \phi(v)|$

Tribonacci 4-graceful labelling of Fork graph is given in the Figure 3.5

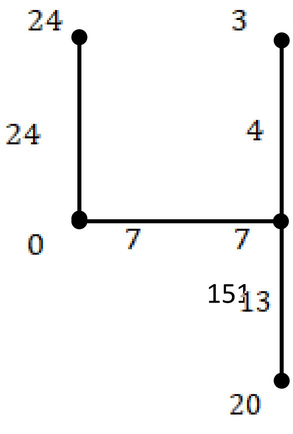

Figure 3.5

Thus ϕ admits Tribonacci graceful labelling for all $r \geq 1$. Therefore Fork graph is Tribonacci arbitrarily graceful graph.

CONCLUSION

In this study, we explored Tribonacci r-graceful labeling for Path graph, Comb graph, Coconut Tree graph, Star graph, Fork graph. The main objective was to determine whether the given graphs admits a Tribonacci r-graceful labeling and analyse their properties in relation to Tribonaaci numbers. This type of labeling has applications in coding theory, cryptography, and communication network design, data structuring and storage, circuit design, scheduling problems, combinatorial design and Graph decomposition.

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