GAUSSIAN TRIBONACCI r-GRACEFUL AND TRIBONACCI PRODUCT CORDIAL LABELING OF PATH RELATED GRAPHS

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Abstract: A Gaussian Tribonacci r-graceful labeling is a specific type of graph labeling where the vertex labels are derived from the Gaussian Tribonacci sequence**.** Let r be any natural number. An injective function $\phi : V(G) \rightarrow \{0, ki, 1, 1+ki, 2, 2+ki, \dots, GT_{q+r-1}\}$ for all k, where GT_{q+r-1} is the $(q+r-1)^{th}$ Gaussian Tribonacci number in the Gaussian Tribonacci sequence is said to be Gaussian Tribonacci r-graceful if the induced edge labeling $\phi^*: E(G) \rightarrow \{GT_r, GT_{r+1}, \dots, GT_{q+r-1}\}$ such that $\phi^*(uv) = |\phi(u) - \phi(v)|$ is bijective. If a graph G admits Gaussian Tribonacci r-graceful labeling, then G is called a Gaussian Tribonacci r-graceful graph. An injective function $\delta : R(G) \rightarrow T_1, T_2, \dots T_m$ is said to be Tribonacci product cordial labeling if the induced function δ^*: B(G)→ {0,1} defined by , $\delta^*(r_i r_j)$ = $((\delta r_i)(\ \delta(r_j))\ (mod 2)$ satisfies the condition | $b\delta^*$ (0) − $b\delta^*$ (1)| ≤ 1. A graph which admits Tribonacci Product cordial labelling is called Tribonacci product cordial graph.

Keywords: Gaussian Tribonacci r-graceful labeling graph, Tribonacci Product Cordial graph, Star graph, Twig graph, Fork graph, Eiffel tower graph, Tribonacci Sequence.

I INTRODUCTION

Graph labeling is an important area in discrete mathematics and combinatorial graph theory, where numbers are assigned to the vertices or edges of a graph according to specific mathematical rules. The concept of graceful labeling, introduced by Rosa in 1967, plays a fundamental role in various applications, including network topology, communication design, circuit layouts, and cryptography. Graceful labeling ensures that the absolute differences between vertex labels along the edges form a unique set, often making it useful for constructing efficient network structures. The Gaussian Tribonacci r-graceful labeling of a graph is an injective assignment of Gaussian Tribonacci numbers to its vertices such that the absolute differences of adjacent vertex labels form a bijective mapping to a set of Tribonacci derived edge labels. If a graph maintains this property for all values of r, it is called Gaussian Tribonacci arbitrarily graceful. The concept of graph labeling was introduced by Rosa in 1967.

Bala et.al., discussed the concept of Tribonacci Product Cordial labeling. An injective function

$\delta : R(G) \rightarrow T_1, T_2, \ldots T_m$ is said to be Tribonacci product cordial labeling if the induced function δ^*: B(G)→ {0,1} defined by , $\delta^*(r_i r_j)$ = $((\delta r_i)(\ \delta(r_j))\ (mod 2)$ satisfies the condition

$|\ b\delta^*(0) - b\delta^*(1)| \leq 1$. A graph which admits Tribonacci Product cordial labelling is called Tribonacci product cordial graph.

II PRELIMINARIES

In this section, we recall some basic definitions related to this paper.

Definition 2.1: A **Graph** G consists of a pair $(V(G), X(G))$ where *V(G)* is a non-empty finite set whose elements are called points or vertices and $X(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called lines or edges of the Graph G.

Definition 2.2: A **Path** is a sequence of vertices where each consecutive pair of vertices is connected by an edge in the Graph.

Definition 2.3: G is called a **labeled graph** if each edge $e = uv$ is given the value $f(uv) = f(u) * f(v)$, where * is a binary operation**.**

Definition 2.4: A **Star graph** S_n is a tree where one vertex is connected to all other vertices.

Definition 2.5: A **Twig graph** T_n is obtained from P_n by attaching exactly two pendent edges to each internal vertices of P_n .

Definition 2.6: The **Fork graph**, sometimes also called the chair graph, is the 5 vertex and 4 edges is also known as 'h graph'.

Definition 2.7: The **Eiffel Tower graph** is the graph on 7 vertices and 7 edges.

Definition 2.8: If we define, for any edge $e = \{u, v\} \in\ E(G)$, the value $\varphi^*(e) = |\varphi(u) - \varphi\ (v)|$ then φ^* is a one-to-one mapping of the set E(G) onto the set $\{1,2, \ldots, n\}$. A graph is called **graceful** if it has a graceful labeling.

Definition 2.9 : The Tribonacci sequence is defined by each term is the sum of the three preceding terms. In other words, The sequence T_n of Tribonacci numbers is defined by third order linear recurrence relation $(n \geq 0)$: $T_{n+3} = T_n + T_{n+1} + T_{n+2}$; $T_0 = 0$, $T_1 + T_2 = 1$ i.e) {0,1,1,2,4,7,13,24,44,81,…} is the **Tribonacci sequence**.

Definition 2.10: An injective $\phi : V(G) \rightarrow \{\ 0,1,2, \ldots, T_{q+r-1}\}$ where T_{q+r-1} is the $(q + r - 1)^{th}$ Tribonacci number in the Tribonacci sequence is said to be Tribonacci r-graceful if the induced edge labeling $\phi^* : E(G) \rightarrow\ T_r, T_{r+1}, T_{q+r-1}$ such that

$\phi^{*}(uv) \rightarrow |\phi(u) - \phi(v)|$ is bijective . if a graph G admits Tribonacci r-graceful labeling, then G is called a **Tribonacci r-graceful graph**.

Definition 2.11: Let G be a graph with p vertices and q edges. Let r be any natural number. An injective function $\phi : V(G) \rightarrow \{0, ki, 1, 1 + ki, 2, 2 + ki, \ldots, GT_{q+r-1}\}$ for all k, where GT_{q+r-1} is the $(q + r - 1)^{th}$ Gaussian Tribonacci number in the Gaussian Tribonacci sequence is said to be Gaussian Tribonacci r-graceful if the induced edge labeling $\phi^{*}: E(G) \rightarrow \{GT_r, GT_{r+1}, \ldots, GT_{q+r-1}\}$ such that $\phi^{*}(uv) = |\phi(u) - \phi(v)|$ is bijective. If a graph G admits Gaussian Tribonacci r-graceful labeling, then G is called a **Gaussian Tribonacci r-graceful graph.**

Definition 2.12 : A binary operation on a graph is called a graph **Product** it is an operation that takes two graphs, G_1 and G_2, and creates a graph H.

Definition 2.13 : A graph G is said to be a **Product Cordial graph** if there exists a function $f: V \rightarrow \{0,1\}$ with each edge assign the label $f(u)f(v)$, such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges respectively labelled with i $(i = 0,1)$.

Definition 2.14 : A function $f : V(G) \rightarrow \{0,1\}$ is said to be **Cordial Labeling** if the induced function $f^{*} : E(G) \rightarrow \{0,1\}$ defined by $f^{*}(uv) \rightarrow | f(u) - f(v)|$ satisfies the conditions $|vf(0) - vf(1)| \leq 1$, as well as $|ef(0) - ef(1)| \leq 1$, where

$vf(0) :=$ number of vertices with label 0,

$vf(1) :=$ number of vertices with label 1,

$ef(0) :=$ number of edges with label 0 ,

$ef(1) :=$ number of edges with label 1.

Definition 2.15 : An injective function $f : V(G) \rightarrow \{T_0, T_1, \cdots, T_n\}$ is said to be **Tribonacci Cordial labeling** if the induced function $f^{*}: E(G) \rightarrow \{0,1\}$ defined by $f^{*}(uv) = (f(u) + f(v)) \pmod 2$ satisfies the condition $|ef(0) - ef(1)| \leq 1$. A graph which admits Tribonacci cordial labeling is called Tribonacci cordial graph.

Definition 2.16: An injective function $\delta : R(G) \longrightarrow \{T_1, T_2, T_3, \ldots, T_n\}$ is said to be Tribonacci product cordial labelling if the induced function $\delta^{*}: B \rightarrow \{0,1\}$ defined by $\delta^{*}(r_i r_j)(\,\delta(r_i)\,\delta(r_j))(mod\ 2)$ satisfies the condition $|B_{\delta^{*}}(0) - B_{\delta^{*}}(1)| \leq 1$. A graph which admits Tribonacci Product cordial labelling is called **Tribonacci Product Cordial graph**.

III GAUSSIAN TRIBONACCI r-GRACEFUL LABELING

Theorem 3.1: The Star graph S_n is Gaussian Tribonacci r-graceful graph for all $n \geq 3$.

Proof: A star graph is a tree with one internal node and multiple leaves.

Let v be the central vertex.

Let $\{v_1, v_2, \ldots, v_n\}$ be the pendant vertices adjacent to v.

The resultant graph is S_n with

$V(S_n) = \{v\} \cup \{v_i / 1 \leq i \leq n-1\}$ and

$E(S_n) = \{vv_i / 1 \leq i \leq n-1\}$.

$|V(S_n)| = p = 3n-4$ and $|E(S_n)| = q = 3n-5$

Define a function $\phi : V(G) \rightarrow \{0, ki, 1, 1+ki, 2, 2+ki, \ldots, GT_{q+r-1}\}$ by

$$\phi(v) = GT_{q+r-1}$$

$$\phi(v_1) = GT_0$$

$$\phi(v_i) = \phi(v) - GT_{q+r-1-(i-1)}, 2 \leq i \leq n.$$

$$\phi^*(uv) = |\phi(u) - \phi(v)|$$

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Example for Gaussian Tribonacci r-graceful labeling for the graph S_8 is given in the Figure 3.1. Edge label calculation for S_8 is given by the following equations,

$\phi^*(uv) = |\phi(u) - \phi(v)|$

$\phi^*(vv_1) = |\phi(v) - \phi(v_1)| = |(81+44i) - 0| = |81+44i| = 81+44i$

$\phi^*(vv_2) = |\phi(v) - \phi(v_2)| = |(81+44i) - (37+20i)| = |44+24i| = 44+24i$

$\phi^*(vv_3) = |\phi(v) - \phi(v_3)| = |(81+44i) - (57+31i)| = |24+13i| = 24+13i$

$\phi^*(vv_4) = |\phi(v) - \phi(v_4)| = |(81+44i) - (68+37i)| = |13+7i| = 13+7i$

$\phi^*(vv_5) = |\phi(v) - \phi(v_5)| = |(81+44i) - (74+40i)| = |7+4i| = 7+4i$

$\phi^*(vv_6) = |\phi(v) - \phi(v_6)| = |(81+44i) - (77+42i)| = |4+2i| = 4+2i$

$\phi^*(vv_7) = |\phi(v) - \phi(v_7)| = |(81+44i) - (79+43i)| = |2+i| = 2+i$

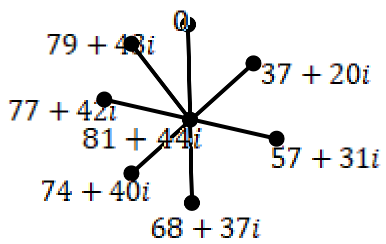

Figure 3.1

Therefore, Star graph S_n admits Gaussian Tribonacci r-graceful labeling for all $n \geq 3$.

Theorem 3.2: The Twig graph T_n is Gaussian Tribonacci r-graceful graph for all $n \geq 3, r \geq 2$.

Proof: Let v_i, $1 \leq i \leq n$ be the vertices of the path P_n.

Let u_i^j, $1 \leq i \leq n$, $j = 1,2$ be the vertices which are attached to the internal vertices of the path P_n.

The resultant graph is T_n with

$V(T_n) = \{\{v_i/1 \leq i \leq n\} \cup \{u_i^1, u_i^2/2 \leq i \leq n-1\}\}$ and

$E(T_n) = \{\{v_i u_i^1, v_i u_i^2/2 \leq i \leq n-1\} \cup \{v_i v_{i+1}/1 \leq i \leq n-1\}\}$ such that

$|V(T_n)| = p = 3n - 4$ and $|E(T_n)| = q = 3n - 5$

Define a function $\phi : V(G) \to \{0, ki, 1, 1 + ki, 2, 2 + ki, \ldots, GT_{q+r-1}\}$ by

$$\phi(v_1) = GT_0$$

$$\phi(v_i) = \phi(v_{i-1}) + (-1)^i GT_{q+r-1-3(i-2)}, 2 \leq i \leq n, r \geq 2$$

$$\phi(u_i^j) = \phi(v_i) - GT_{q+r-1-(3i-j-3)}, 2 \leq i \leq n, j = 1,2, r \geq 2$$

$$\phi^*(uv) = |\phi(u) - \phi(v)|$$

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Hence the Twig graph T_n is Gaussian Tribonacci r-graceful for all $n \geq 3, r \geq 2$.

Example for the Gaussian Tribonacci 5-graceful labeling for the graph T_{11} is shown in Figure 3.2.

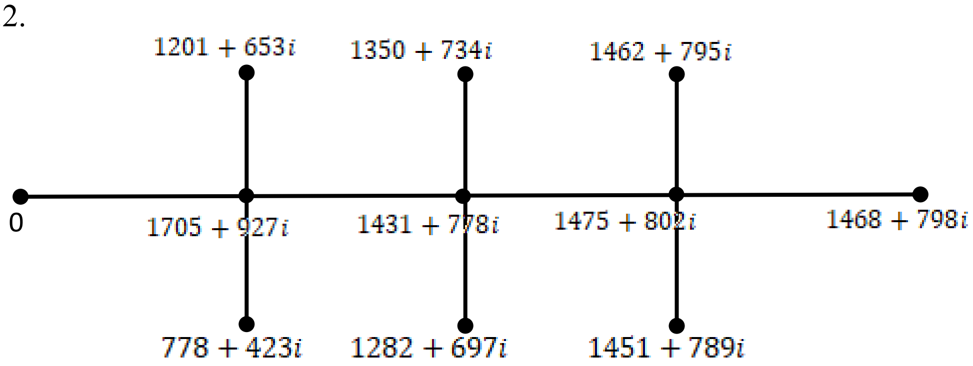

Figure 3.2

Edge label calculation is given by the following equations

$\phi^*(uv) = |\phi(u) - \phi(v)|$

$\phi^*(v_1 v_2) = |\phi(v_1) - \phi(v_2)| = |0 - (1705+927i)| = |-1705-927i| = 1705+927i$

$\phi^*(v_2 v_3) = |\phi(v_2) - \phi(v_3)| = |(1705 + 927i) - (1431 + 778i)| = |274 + 149i|$

$= 274 + 149i$

$\phi^*(v_3v_4) = |\phi(v_3) - \phi(v_4)| = |(1431 + 778i) - (1475 + 802i)| \quad = |-44 - 24i|$

$= 44 + 24i$

$\phi^*(v_4v_5) = |\phi(v_4) - \phi(v_5)| = |(1475 + 802i) - (1468 + 798i)| \quad = |7 + 4i|$

$= 7 + 4i$

$\phi^*(v_2u_2^1) = |\phi(v_2) - \phi(u_2^1) = |(1705 + 927i) - (1201 + 653i)| \quad = |504 + 274i|$

$= 504 + 274i$

$\phi^*(v_3u_3^1) = |\phi(v_3) - \phi(u_3^1) = |(1431 + 778i) - (1350 + 734i)| \quad = |81 + 44i|$

$= 81 + 44i$

$\phi^*(v_4u_4^1) = |\phi(v_4) - \phi(u_4^1) = |(1475 + 802i) - (1462 + 795i)| \quad = |13 + 7i|$

$= 13 + 7i$

$\phi^*(v_2u_2^2) = |\phi(v_2) - \phi(u_2^2) = |(1705 + 927i) - (778 + 423i)| \quad = |927 + 504i|$

$= 927 + 504i$

$\phi^*(v_3u_3^2) = |\phi(v_3) - \phi(u_3^2) = |(1431 + 778i) - (1282 + 697i)| \quad = |149 + 81i|$

$= 149 + 81i$

$\phi^*(v_4u_4^2) = |\phi(v_4) - \phi(u_4^2) = |(1475 + 802i) - (1451 + 789i)| \quad = |24 + 13i|$

$= 24 + 13i.$

Therefore, Twig graph T_n admits Gaussian Tribonacci r-graceful labeling for all $n \geq 3, r \geq 2$.

Thus ϕ admits Gaussian Tribonacci r-graceful labelling.

Theorem 3.3: The Fork graph is Gaussian Tribonacci r-graceful labeling for all $r \geq 1$.

Proof: Let $V(G) = \{v_i / 1 \leq i \leq 5\}$

Let $E(G) = \{e_i / 1 \leq i \leq 4\}$.

Define a function $\phi : V(G) \to \{0, ki, 1, 1 + ki, 2, 2 + ki, \ldots, GT_{q+r-1}\}$ by

$$
\begin{aligned}
\phi(v_1) &= GT_0 \\
\phi(v_2) &= GT_{r+1} \\
\phi(v_3) &= GT_{q+r-1} \\
\phi(v_4) &= \phi(v_2) - GT_r \\
\phi(v_5) &= \phi(v_2) + GT_{r+2} \\
\phi^*(uv) &= |\phi(u) - \phi(v)|
\end{aligned}
$$

Hence the condition $|e_f(0) - e_f(1)| \leq 1$ is satisfied.

Thus ϕ admits Tribonacci graceful labelling for all r.

The Gaussian Tribonacci r –graceful labeling for the Fork graph is given in the Figure 3.3.

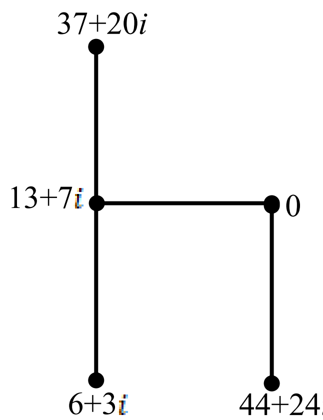

Figure 3.3

Therefore, Fork graph admits Gaussian Tribonacci r-graceful labeling for all $r \geq 1$.

IV TRIBONACCI PRODUCT CORDIAL LABELING

Theorem 4.1: The Star graph S_n is Tribonacci product cordial graph.

Proof: Define the function $\delta: R \rightarrow \{T_1, T_2, T_3, \dots, T_n\}$ to label the vertex as follows:

$$\delta(v_1) = T_1$$

$$\delta(v_i) = T_{i+1}, 1 \leq i \leq n.$$

To obtain the edge labels, define the induced function $\delta^*: B \rightarrow \{0,1\}$ defined by

$$\delta^*(r_i r_j) = (\ \delta(r_i)\ \delta(r_j))(mod\ 2).$$

Thus using the induced function the edges receive the labels as follows:

$$\delta^*(r_i r_{i+1})\ = 1, i = 1,2$$

$$\delta^*(r_i r_{i+2})\ = 0, i = 2$$

$$\delta^*(r_i r_{i+1})\ = 1, i = 4$$

Hence the condition $|B_{\delta^*}(0) - B_{\delta^*}(1)| \leq 1$ is satisfied.

Example for the Tribonaacci product cordial labeling for the graph S_5 is shown in the Figure 4.1.

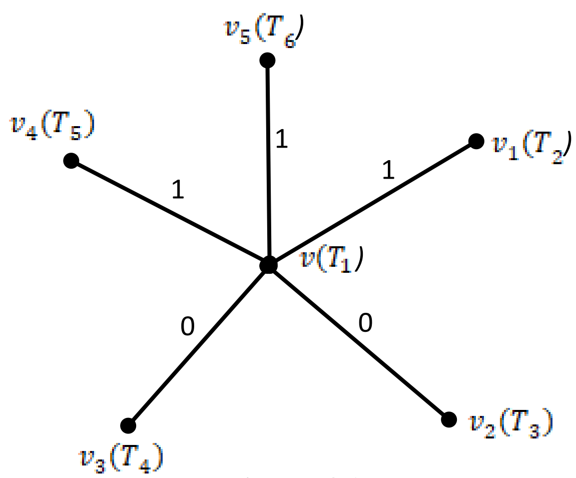

Figure 4.1

Therefore, the Fork graph admits Tribonacci product cordial labelling.

Theorem 4.2: The Eiffel tower graph is Tribonacci product cordial graph.

Proof: From the structure of Eiffel tower graph it is clear that is has 7 vertices and 7 edges.

Define the function $\delta: R \rightarrow \{T_1, T_2, T_3, \ldots, T_n\}$ to label the vertex as follows:

$$\delta(v_i) = T_{i+3}, \quad 1 \leq i \leq 3$$

$$\delta(v_4) = T_3$$

$$\delta(v_5) = T_7$$

$$\delta(v_i) = T_{i-5} \quad, i = 6, 7$$

To obtain the edge labels, define the induced function $\delta^*: B \rightarrow \{0,1\}$ defined by

$$\delta^*(r_i r_j) = (\ \delta(r_i)\ \delta(r_j))(mod\ 2).$$

Thus using the induced function the edges receive the labels as follows:

$$\delta^*(r_i r_{i+1}) = 1, i = 1,2$$

$$\delta^*(r_i r_{i+2}) = 0, i = 2$$

$$\delta^*(r_i r_{i+1}) = 1, i = 4$$

Hence the condition $|B_{\delta^*}(0) - B_{\delta^*}(1)| \leq 1$ is satisfied.

The Tribonacci product cordial labeling of Eiffel Tower graph is shown in the Figure 4.2.

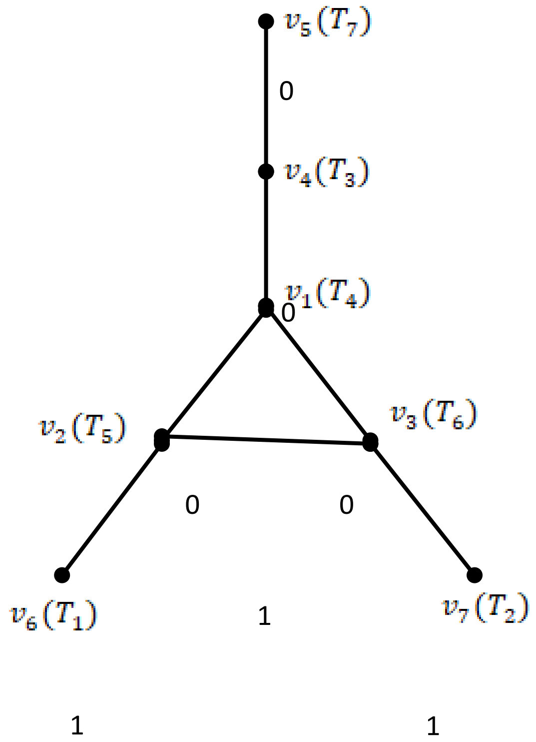

Figure 4.2

Therefore, the Eiffel tower graph admits Tribonacci product cordial labelling.

Theorem 4.3: The Fork graph is Tribonacci product cordial labeling.

Proof: From the structure of Fork graph it is clear that is has 5 vertices and 4 edges.

Define the function $\delta: R \to \{T_1, T_2, T_3, \dots, T_n\}$ to label the vertex as follows:

$$\delta(v_i) = T_i, i = 1,2$$

$$\delta(v_i) = T_{i+2}, i = 3$$

$$\delta(v_i) = T_{i-1}, i = 4,5$$

To obtain the edge labels, define the induced function $\delta^*: B \to \{0,1\}$ defined by

$\delta^*(r_i r_j) = (\ \delta(r_i)\ \delta(r_j))(mod\ 2)$. Thus using the induced function the edges receive the labels as follows:

$$\delta^*(r_i r_{i+1}) = 1, i = 1,2$$

$$\delta^*(r_i r_{i+2}) = 0, i = 2$$

$$\delta^*(r_i r_{i+1}) = 1, i = 4$$

Hence the condition $|B_{\delta^*}(0) - B_{\delta^*}(1)| \le 1$ is satisfied.

The Tribonacci product cordial labeling of Fork graph is shown in the Figure 4.3.

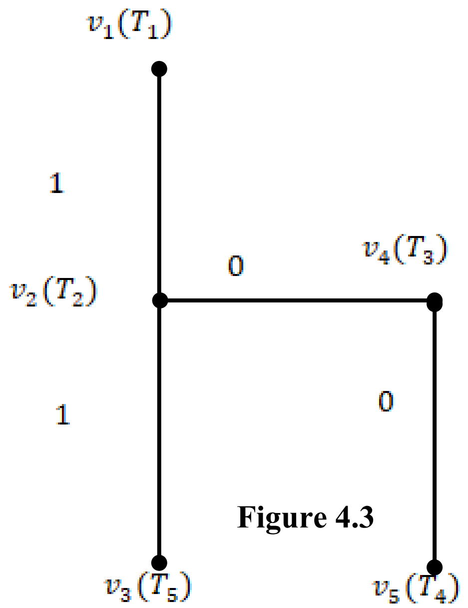

Figure 4.3

Therefore, the Fork graph admits Tribonacci product cordial labelling.

CONCLUSION

In this paper Gaussian Tribonacci r-Graceful Labeling and Tribonacci Product Cordial Labeling, both extending classical labeling contribute to the understanding of balance and symmetry in graphs, with potential applications in network theory, coding, and optimization.Our results showed that Gaussian Tribonacci r-Graceful Labeling successfully extends r-graceful labeling to Tribonacci-based graphs, while Tribonacci Product Cordial Labeling offers a balanced approach for labeling graphs based on the product of adjacent labels. These findings deepen our understanding of graph labelings and open up avenues for future research in graph theory and applied fields. Future work could include investigating these labelings in more complex graph classes and developing efficient algorithms for large-scale networks. Overall, this research enhances the theoretical framework for Tribonacci-based labelings and suggests new possibilities for their application in various domains.

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